

Minimal Self-Joinings, Bounded Constructions, and Weak Closure of Ergodic Actions.

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For a weakly mixing bounded rank-one construction it is proved the disjointness of its powers. For non-rigid constructions we get minimal self-joinings. Examples of non-mixing rank one actions with explicit weak closure are proposed.

1 Introduction

Consider rank-one measure-preserving transformations of a Probability space (X, μ) . In connection with Thouvenot's question on minimal self-joinings (MSJ) of mildly mixing rank-one transformations, for a class of transformations (mildly mixing bounded constructions) we prove the property MSJ. This class includes well-known classical and modified Chacon's transformations (see[2]).

The "springs" of our proof are similar to ones of [2], although we do not consider generic points, and never use Birkhoff's ergodic theorem. Weak limits are our tools. In operator terms the method is following. Let P be an indecomposable Markov operator commuting with a rank-one transformation T (P is extreme point of Markov centralizer of T). If $P \neq T^i$ for all $i \in \mathbf{Z}$, then there is a sequence n_j such that $T^{n_j} \rightarrow aP + (1-a)P'$, where $a \geq \frac{1}{2}$, P' is some Markov operator. In fact there is a sequence of sets Y_j such that $\mu(Y_j) \geq \frac{1}{2}$ and

$$\mathbf{1}_{Y_j} T^{n_j} \rightarrow aP, \quad a \geq \frac{1}{2}.$$

Using non-rigidity of T find a sequence $C_j \subset Y_j$ for which

$$\mathbf{1}_{C_j} T^{n_j} \rightarrow cT^m P, \quad c > 0, \quad m \neq 0.$$

But from $C_j \subset Y_j$ we have $\mathbf{1}_{C_j} T^{n_j} \rightarrow cP$ as well. This implies

$$P = T^m P.$$

If T is mildly mixing, then T^m ($m \neq 0$) is ergodic, hence, $P = \Theta$ – the ortho-projector to constant functions in $L_2(X, \mu)$. We get the triviality of Markov centralizer. It is the set of all convex sums of the operators T^i and Θ , consequently, T has MSJ.

In [2] the authors used a similar trick: they proved for a non-trivial self-joining ν of Chacon's map the marginal invariance:

$$\nu = (Id \times T)\nu,$$

so $\nu = \mu \times \mu$.

We give new examples of non-mixing rank one actions with explicit weak closure that gives minimal self-joinings. An action of such kind appeared in arXiv:1108.0568: for double staircase transformation T the weak closure of its powers $Lim(T)$ is

$$\{\Theta, 2^{-m}T^n + (1 - 2^{-m})\Theta : m \in \mathbf{N}, n \in \mathbf{Z}\}.$$

A non-simple description of power weak limits for Chacon's transformation is given in [12].

In this note we show that $\text{Lim}(T)$ for so-called stochastic Chacon's transformation (see [8]) is (a.s.)

$$\{\Theta, P^m P^{*n} T^k : m, n \in \mathbf{N}, k \in \mathbf{Z}, m + n \neq 0, P = aI + (1 - a)T\}.$$

For bounded staircase flow T_t we get

$$\text{Lim}(T_t) = \{\Theta, T_a \prod_{i \in S} P_{m_i} : a \in \mathbf{R}, S \subset \mathbf{N}, |S| < \infty, P_m = \int_0^m T_t dt\}.$$

2 Rank-one constructions

A rank-one construction is determined by an integer h_1 , a cut-sequence r_j and a spacer sequence \bar{s}_j

$$\bar{s}_j = (s_j(1), s_j(2), \dots, s_j(r_j - 1), s_j(r_j)).$$

We recall its definition. Let a transformation T is defined on the step j as a shift on a collection of disjoint sets (intervals)

$$E_j, TE_j T^2 E_j, \dots, T^{h_j} E_j.$$

Cut E_j into r_j sets (subintervals) of the same measure

$$E_j = E_j^1 \sqcup E_j^2 \sqcup E_j^3 \sqcup \dots \sqcup E_j^{r_j},$$

then for all $i = 1, 2, \dots, r_j$ we consider so-called columns

$$E_j^i, TE_j^i, T^2 E_j^i, \dots, T^{h_j} E_j^i.$$

Adding $s_j(i)$ spacers over a i -th column we obtain "column + spacers"

$$E_j^i, TE_j^i T^2 E_j^i, \dots, T^{h_j} E_j^i, T^{h_j+1} E_j^i, T^{h_j+2} E_j^i, \dots, T^{h_j+s_j(i)} E_j^i$$

(all intervals are disjoint). For all $i < r_j$ we "stack"

$$TT^{h_j+s_j(i)} E_j^i = E_j^{i+1}.$$

Now one obtains a tower

$$E_{j+1}, TE_{j+1} T^2 E_{j+1}, \dots, T^{h_{j+1}} E_{j+1},$$

where

$$\begin{aligned} E_{j+1} &= E_j^1, \\ T^{h_{j+1}} E_{j+1} &= T^{h_j+s_j(r_j)} E_j^{r_j}, \\ h_{j+1} + 1 &= (h_j + 1)r_j + \sum_{i=1}^{r_j} s_j(i). \end{aligned}$$

Bounded constructions. Following [10] consider constructions assuming that all $s_j(i)$, r_j are bounded: $0 \leq s_j(i) < s$, $2 < r_j < r$.

(A reader can slightly generalize the above notion, only requiring for all j bounded "derivatives": $\max_{1 \leq i < r_j} |s_j(i+1) - s_j(i)| < s$, $r_j < r$.)

If $s_j(r_j) = 0$ for all j , we get a "strongly bounded construction", which is as a rule similar to modified Chacon's transformation.

Examples.

Odometers. For instance, $r_j = 5$, $\bar{s}_j = (2, 2, 2, 2, 0)$ for all j .

Classic Chacon's map: $r_j = 2$, $\bar{s}_j = (0, 1)$ for all j . It is not mixing, that is clear because of the following weak convergence

$$2T^{-h_j} \rightarrow \sum_{i=0}^{\infty} \left(\frac{T}{2}\right)^i.$$

Modified Chacon's map ([2]): $r_j = 3$, $\bar{s}_j = (0, 1, 0)$ for all j . Now

$$T^{-h_j} \rightarrow \frac{I + T}{2}.$$

3 Joinings

A *self-joining* (of order 2) is defined to be a $T \times T$ -invariant measure ν on $X \times X$ with the marginals equal to μ :

$$\nu(A \times X) = \nu(X \times A) = \mu(A).$$

A joining ν is called ergodic if the dynamical system $(T \times T, X \times X, \nu)$ is ergodic. The measures $\Delta^i = (Id \times T^i)\Delta$ defined by the formula

$$\Delta^i(A \times B) = \mu(A \cap T^i B)$$

are referred to as *off-diagonals measures* (for $i \neq 0$). If T is ergodic, then Δ^i are ergodic self-joinings.

We say that T has **minimal self-joinings** of order 2 (and we write $T \in MSJ(2)$) if T has no ergodic joinings except $\mu^{\otimes 2} = \mu \times \mu$ and Δ^i .

The notion of MSJ (of all orders) has been introduced by D. Rudolph [1] (see also [7]). In [2] the authors proved MSJ for modified Chacon's automorphism. It is well-known fact that $MSJ(2)=MSJ$ for non-mixing maps (Glasner, Host, Rudolph, Ryzhikov).

The property of minimal self-joinings implies **mild mixing**. We recall that an automorphism T is *mildly mixing* if for any set A , $0 < \mu(A) < 1$,

$$\limsup_j \mu(A \cap T^i A) < \mu(A).$$

An automorphism T is mildly mixing iff it has no rigid factors (S is rigid, if there is a sequence $m_j \rightarrow \infty$ such that $S^{k_j} \rightarrow Id$).

An automorphism T is *partially mixing*, if for some $\alpha \in (0, 1]$ and for all measurable sets A, B

$$\liminf_i \mu(A \cap T^i B) \geq \alpha \mu(A) \mu(B).$$

In [6] the authors proved minimal self-joinings for partially mixing rank one transformations. The property of partial mixing implies mildly mixing.

J.-P. Thouvenot conjectured that all **bounded mildly mixing constructions have minimal self-joinings**. We give a positive answer (generalizing [2] and some results from [5]).

THEOREM. *Let T be a bounded non-rigid construction. If it is totally ergodic (all non-zero powers are ergodic), then T has minimal self-joinings.*

4 Ergodic limits of off-diagonals are trivial

Given an ergodic self-joining ν of a rank one transformation T , is there a sequence n_j such that for all measurable A, B

$$\nu(A \times B) = \lim_{j \rightarrow \infty} \mu(T^{n_j} A \cap B) \quad ?$$

This question is due to J. King.

If such a sequence n_j exists, we call ν a limit of off-diagonals and write

$$\Delta_{T^{n_j}} \rightarrow \nu.$$

LEMMA *Let T be non-rigid totally ergodic bounded construction, $m_j \rightarrow \infty$ and $\Delta_{T^{m_j}} \rightarrow \nu$. If ν is ergodic, then $\nu = \mu \times \mu$.*

Proof. We find $p = p(i)$ such that $h_p \leq m_j < h_{p+1}$. Now we consider our construction on the step p . Let's remark that $p = p(i) \rightarrow \infty$ and $rh_p > h_{p+1}$ for all (large) p . The spacers over the last column form the roof over $(p+1)$ -th tower. The transformation T is non-rigid, hence the roof have to be (asymptotically) non-flat. This implies the following:

$$T^{m_j} C_j \rightarrow cP,$$

and for some Markov operator Q commuting with T we have

$$P = (a_0 I + a_1 T + \dots + a_{r-1} T^{r-1})Q,$$

where at list two coefficients, say a_n, a_{n+k} , $k > 0$, have to be both non-zero. Let us rewrite this in joining terms:

$$\Delta_{T^{m_j}} \rightarrow \nu = \sum_{k=0}^{r-1} a_k (Id \times T^k) \eta.$$

Thus, ν and $(Id \times T^k)\nu$ are not disjoint. Assuming ν to be ergodic we get

$$\nu = (Id \times T^k)\nu.$$

From the ergodicity of T^k it follows that

$$\nu = \mu \times \mu$$

(in operator terms we say: $T^k P = P$ implies $P = \Theta$.) Indeed,

$$\begin{aligned} \nu(A \times B) &= \int_{X \times X} \chi_A \otimes \left(\frac{1}{N} \sum_{d=1}^N T^{dk} \chi_B \right) d\nu = \\ &= \lim_N \int_{X \times X} \chi_A \otimes \left(\frac{1}{N} \sum_{d=1}^N T^{dk} \chi_B \right) d\nu = \mu(B) \int_{X \times X} \chi_A \otimes 1 d\nu = \mu(A)\mu(B). \end{aligned}$$

5 Joinings as Local Limits

Any ergodic self-joining ν of a rank one transformation T is a partial limit of off-diagonals: there is a sequence n_j such that

$$\Delta_{T^{n_j}} \rightarrow \frac{1}{2}\nu + \frac{1}{2}\nu'.$$

In fact (see [11]), for some $\delta \leq \frac{1}{2}$ there is a sequence of sets Y_j in the form

$$Y_j = \bigcup_{\delta h_j < k < h_j} T^k E_j$$

and a sequence $\{n_j\}$, $n_j \approx (1 - \delta)h_j$, such that

$$\nu(A \times B) = \lim_{j \rightarrow \infty} \frac{1}{\mu(Y_j)} \mu(T^{n_j} A \cap B \cap Y_j),$$

equivalently,

$$\frac{1}{\mu(Y_j)} \mathbf{1}_{Y_j} T^{n_j} \rightarrow P.$$

Such limits we call local. The following lemma shows that sometimes certain local limits become global.

LEMMA 1. *Let $\mathbf{1}_{Y_j} T^{n_j} \rightarrow (1 - \delta)P$, $h_j \leq n_j < h_{j+1}$. We represent*

$$n_j = qh_j + s_j(1) + s_j(2) + \dots + s_j(q) + m_j, \quad 0 \leq m_j < h_j.$$

If $\frac{m_j}{h_j} \rightarrow 0$ (or $\frac{h_j - m_j}{h_j} \rightarrow 0$), then $T^{m_j} \rightarrow P$ (or $T^{m_j - h_j} \rightarrow P$).

Proof. Let C_j^1 denote the first column of j -tower. We see that

$$\mu(C_j^1) > \frac{1}{r}, \quad T^{n_j} C_j^1 \subset Y_j, \quad \mu(T^{n_j} C_j^1 \Delta C_j^{q+1}) \rightarrow 0.$$

We get

$$\begin{aligned} \nu(A \times B) &= \lim_{j \rightarrow \infty} \frac{1}{\mu(Y_j)} \mu(T^{n_j} A \cap B \cap Y_j) = \\ &= \lim_{j \rightarrow \infty} \frac{1}{\mu(C_j^1)} \mu(T^{n_j} A \cap T^{n_j} C_j^1 \cap B) = \\ &= \lim_{j \rightarrow \infty} \frac{1}{\mu(C_j^1)} \mu(T^{m_j} A \cap C_j^{q+1} \cap B) = \lim_{j \rightarrow \infty} \mu(T^{m_j} A \cap B). \end{aligned}$$

6 On local "breaking" of non-trivial self-joinings

Assume T satisfies the conditions of Theorem 1. Let's find sequences of sets $C_j', C_j'' \subset Y_j$ such that

$$\mu(C_j') \approx \mu(C_j'') \rightarrow c > 0, \quad \mathbf{1}_{C_j'} \circ T^{n_j} \rightarrow cP, \quad \mathbf{1}_{C_j''} \circ T^{n_j} \rightarrow cP,$$

and in addition for fixed $m > 0$ and some sequence m_j , $|m_j| < h_j$, let

$$\mathbf{1}_{C_j'} \circ T^{n_j} \approx_w \mathbf{1}_{C_j'} \circ T^{m_j}, \quad (*)$$

$$\mathbf{1}_{C_j''} \circ T^{n_j} \approx_w \mathbf{1}_{C_j''} \circ T^{m_j+m}, \quad (**)$$

$$\mathbf{1}_{C_j'} \circ T^{m_j} \approx_w \mathbf{1}_{C_j''} \circ T^{m_j}. \quad (***)$$

If such C_j', C_j'' are found, then

$$cP \approx_w \mathbf{1}_{C_j''} \circ T^{m_j+m} \approx_w \mathbf{1}_{C_j'} \circ T^{m_j+m} \approx_w cT^m P, \quad P = T^m P, \quad P = \Theta.$$

Repeat the above in joining terms : for all measurable A, B

$$\nu(A \times B) = \lim_{j \rightarrow \infty} \mu(T^{n_j} A \cap B \cap C_j'') / \mu(C_j'') =_{(**)} \lim_{j \rightarrow \infty} \mu(T^{m_j+m} A \cap B \cap C_j'') / \mu(C_j'') =_{(***)}$$

$$\lim_{j \rightarrow \infty} \mu(T^{m_j} T^m A \cap B \cap C_j') / \mu(C_j') =_{(*)} \lim_{j \rightarrow \infty} \mu(T^{n_j} T^m A \cap B \cap C_j') / \mu(C_j') = \nu(T^m A \times B).$$

Thus, $\nu = (T^m \times Id)\nu$, $\nu = \mu \times \mu$.

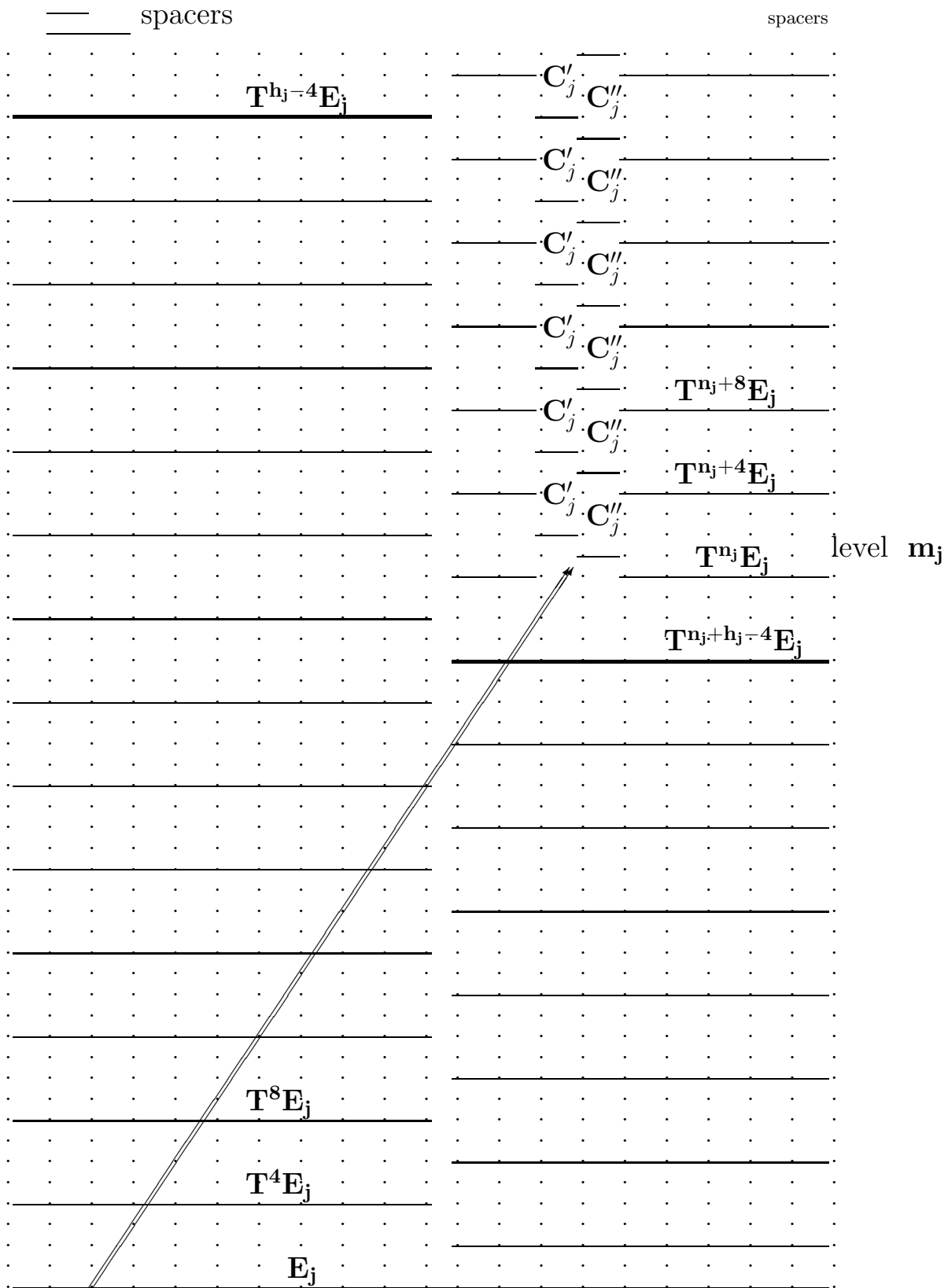
How to find C_j', C_j'' ? Consider minimal i for which $s_j(i) \neq s_j(i+1)$. Without loss of generality suppose $i < 0.4r_j$.

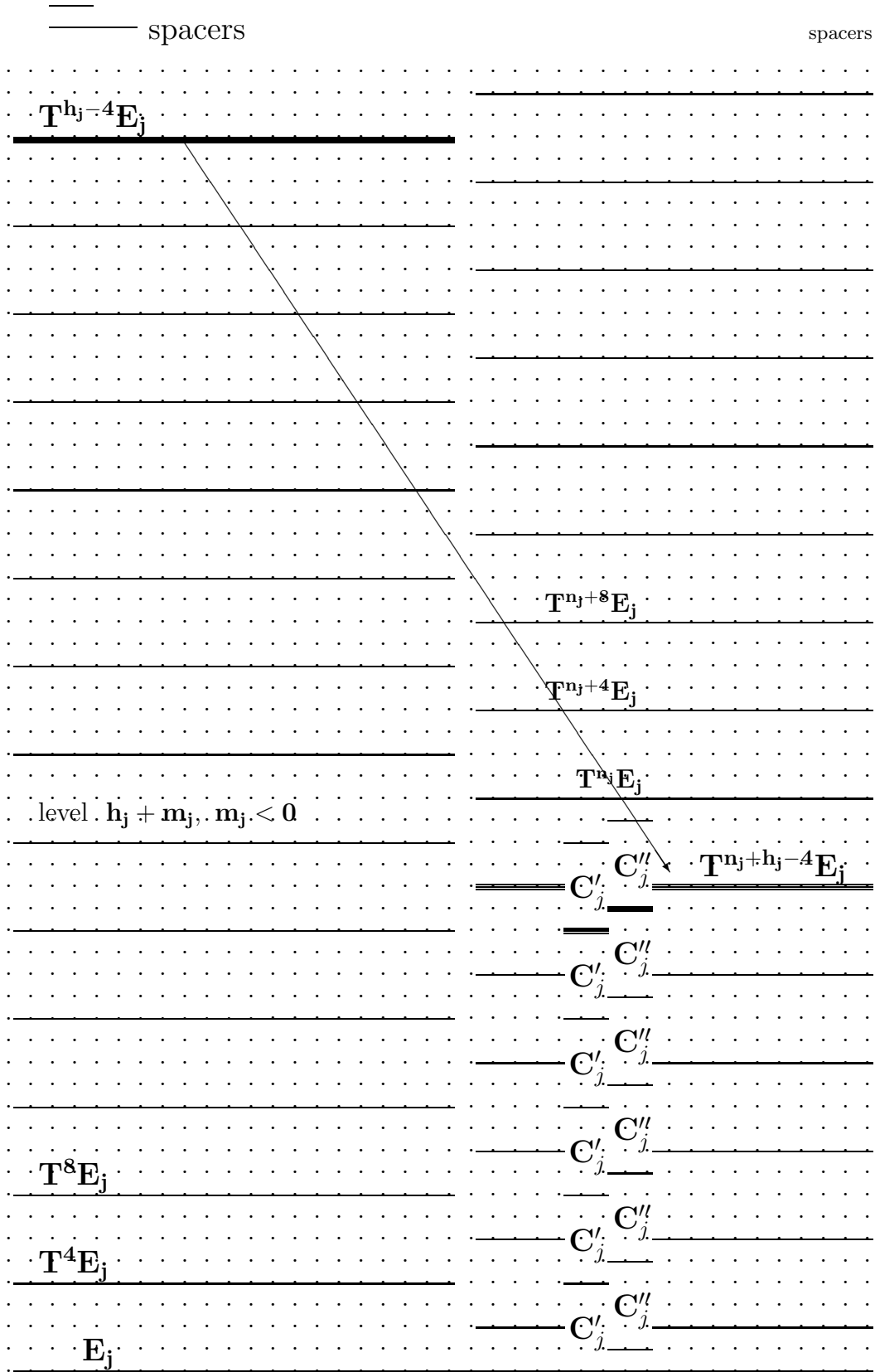
In order to have this, instead of the sequence $\{j\}$ (of steps) we may consider the subsequence $\{Nj\}$, where N is fixed. Now

$$\tilde{r}_j := \prod_{k=(j-1)N+1}^{jN} r_k.$$

Let's look at Chacon's map spacer sequence $\bar{s}_j = (0, 1, 0)$. Setting $j := 2j$, we get new spacer sequence $\bar{\bar{s}}_j = (0, 1, 0, 0, 1, 1, 0, 1, 1)$. Generally, given construction T , if for all N we cannot find i s.th. $s_j(i) \neq s_j(i+1)$, then we see flat roofs of towers, hence, T have to be rigid. It is not our case. Thus, we get for some N for all (new) j a desired integer $i = i(j)$ satisfied $s_j(i) \neq s_j(i+1)$. If N is sufficiently large, we find $i < 0.4r_j$.

If the image under T^{n_j} of the left half of E_j (and the left part of spacers) is located far away from the top and bottom of the tower, then we easy find C_j', C_j'' as below. Let's remark that in the case of the flat image at the top of the tower, we are looking for C_j', C_j'' at the bottom (and vice versa, respectively).





If the image at the top of the tower is flat, we are looking for C'_j, C''_j at the bottom

If the mentioned image of E_j is close to the top, or to the bottom, we use Lemma 1. This finish the proof of Theorem 1.

7 Bounded constructions and disjointness of powers

In [10] J. Bourgain proved that bounded constructions satisfied the Moebius orthogonality property. This property is a consequence of MSJ (in fact, of the disjointness of T^q and T^p , $q \neq p$).

THEOREM 2. *If a bounded construction T is weakly mixing and $q \neq p$, then T^q and T^p are disjoint.*

Proof. If roofs over last columns are asymptotically non-flat, a weakly mixing construction is not rigid, hence, it has MSJ (Theorem 1). If a flatness appears, this means that our spacer sequence is following: for some sequence of integer intervals $[\alpha_t, \beta_t]$ ($\beta_t - \alpha_t \rightarrow \infty$)

$$\bar{s}_j = (s_t, s_t, \dots, s_t, s_t, 0), \quad \forall j \in [\alpha_t, \beta_t]$$

("0"-s are necessary in this situation). From the weak mixing property, from time to time we see a break in the mentioned behavior (our construction is not an odometer!). Thus, infinitely many times we meet non-flat roofs over last columns. Given $\varepsilon > 0$, it is not hard to get for all $k < \varepsilon^{-1}$ the following weak limits:

$$T^{kn_j} \rightarrow (1 - k\varepsilon)I + k\varepsilon P,$$

where $P = \sum_{i \geq 0}^\infty a_i T^i$, $\sum_{i \geq 0}^\infty a_i = 1$, and $a_m > 0$ for some $m > 0$.

If

$$T^q J = J T^p$$

for an operator $J : L_2(\mu) \rightarrow L_2(\mu)$, then

$$((1 - q\varepsilon)I + q\varepsilon P)J = J((1 - p\varepsilon)I + p\varepsilon P). \quad (*)$$

Assume J to be an indecomposable Markov operator (indecomposable in the convex set of Markov operators intertwining T^q and T^p , or, in other words, the corresponding joining is ergodic). All operators $T^n J, J T^m$ are indecomposable as well. Let $q < p$, from (*) it follows $J = J T^{m'}$ for some $m' > 0$ (again a marginal invariance of a joining!). The ergodicity of $T^{m'}$ implies $J = \Theta$. Thus, T^q and T^p are disjoint.

8 Explicit Weak Closure of Actions

Stochastic Chacon's map. Fixing h_1 , cut-numbers $r_j \rightarrow \infty$, consider the set of all constructions with spacer sequences $s_j(i) \in \{0, 1\}$. Let's equip this ensemble by Bernoulli measure of type $(a, 1 - a)$. I.e., in a random fashion, with probability $1 - a$ we stack one spacer over a column. Denote $P = aI + (1 - a)T^{-1}$.

THEOREM 3. *For stochastic Chacon's maps T almost surely*

$$\text{Lim}(T) = \{\Theta, P^m P^{*n} T^k : m, n \in \mathbf{N}, k \in \mathbf{Z}, m + n \neq 0\}.$$

COROLLARY. $\text{Lim}(T) = \{\Theta, T^k P^n : n = 0, 1, 2, \dots, n > 0, k \in \mathbf{Z}\}$ as $a = 0.5$.

LEMMA. Let $T^{n_j} \rightarrow Q$, $n_j > 0$. Represent $n_j = q_j h_{p(j)} + m_j$, where $h_{p(j)} \leq n_j < h_{p(j)+1}$. $0 \leq m_j \leq h_{p(j)}$, $q_j \leq r_{p(j)}$. If $q_j \rightarrow \infty$ and $(r_{p(j)} - q_j) \rightarrow \infty$, then $Q = \Theta$.
The proof of this lemma uses the following facts: For almost all T for any $q > 0$

$$T^{qh_j} \rightarrow P^q$$

and

$$P^q \rightarrow \Theta, \text{ as } q \rightarrow \infty,$$

moreover, $T^n P^q \rightarrow \Theta$ uniformly with respect to n .

Assume that q_j (or $r_{p(j)} - q_j$) to be bounded. If $\varepsilon h_{p(j)} < |m_j| < (1 - \varepsilon) h_{p(j)}$, then $Q = \Theta$. Let

$$\frac{m_j}{h_{p(j)}} \rightarrow 0.$$

For all i , $0 < i < r_{p(j)} - q + 1$

$$T^{m_j - s_j(i) - s_j(i+1) - \dots - s_j(i+q-1)} \approx_w Q,$$

$$T^{n_j} \approx_w \frac{1}{r_{p(j)}} \sum_{i=1}^{r_{p(j)} - q} T^{-s_j(i) - s_j(i+1) - \dots - s_j(i+q-1)} T^{m_j} \approx_w P^q T^{m_j}.$$

Thus, $Q = P^q \lim_j T^{m_j}$, then we get

$$Q = P^k P^{*n} Q''$$

and so on. Note that P^* appears in case of $\frac{m_j}{h_{p(j)}} \rightarrow 1$. Iterating, if we cannot stop by $Q''' \dots''' = T^m$, then $P = \Theta$ ($P^k P^{*n} T^m \rightarrow \Theta$ as $k + n \rightarrow \infty$ uniformly with respect to m).

Bounded staircase flow T_t . Fix $h_1 \in \mathbb{R}^+$ and cut-sequence $r_j \rightarrow \infty$. Define spacers by $s_j(i) = \frac{i-1}{r_j}$ ($1 \leq i \leq r_j$).

We have

$$T_{mh_j} \rightarrow P_m := \int_{-m}^0 T_t dt = T_{-m} P_m^*.$$

THEOREM. For bounded staircase flow T_t

$$\text{Lim}(T_t) = \{\Theta, T_a \prod_{m \in M} P_m : a \in \mathbb{R}, M \subset \mathbb{N}, |M| < \infty\}.$$

LEMMA. Let $T^{t_j} \rightarrow Q$, $h_{p(j)} \leq t_j < h_{p(j)+1}$. Represent $t_j = q_j h_{p(j)} + m_j$, where $0 \leq m_j \leq h_{p(j)}$, $q_j \leq r_{p(j)}$. If $q_j \rightarrow \infty$ and $(r_{p(j)} - q_j) \rightarrow \infty$, then $Q = \Theta$.

The proof of the lemma is an exercise.

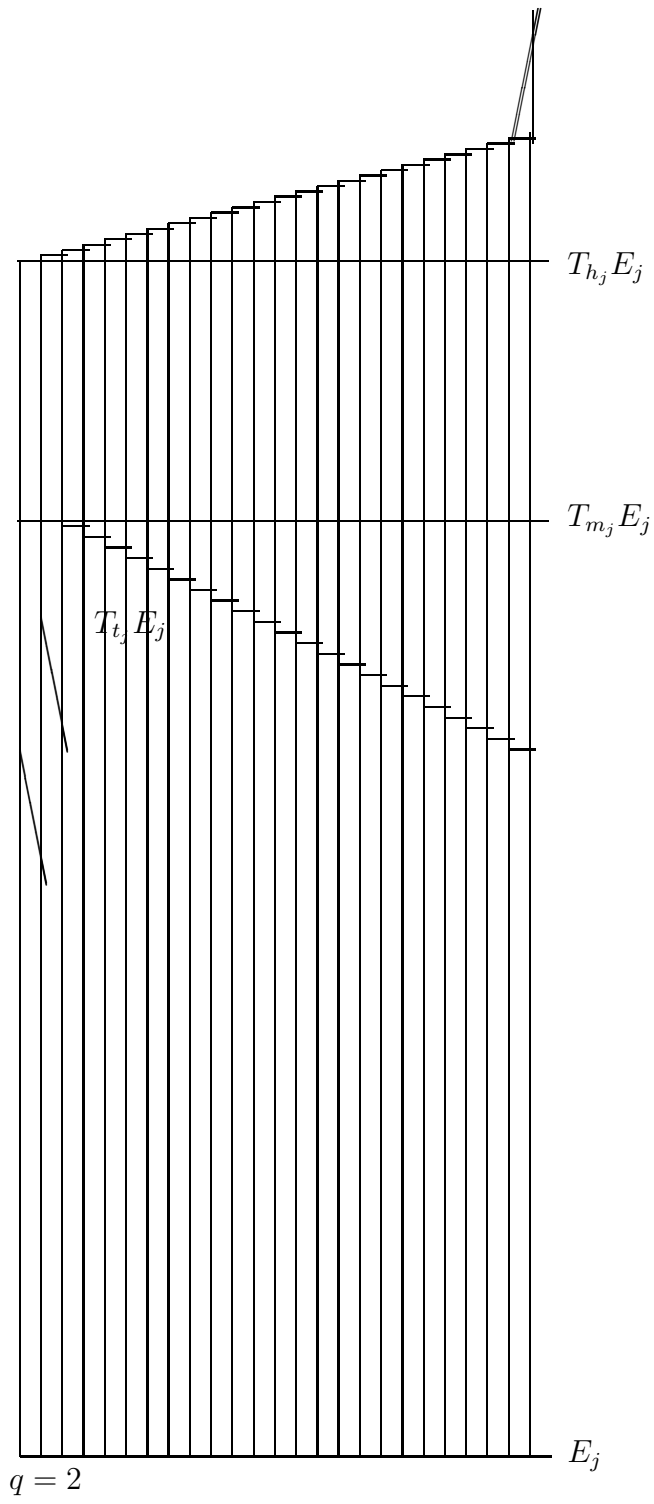
Let $T_{t_j} \rightarrow Q \neq \Theta$, then Lemma asserts that there is q such that $t_j = q h_{p(j)} + m_j$. Again we have

$$\frac{m_j}{h_{p(j)}} \rightarrow 0 \text{ or } \frac{h_{p(j)} - m_j}{h_{p(j)}} \rightarrow 1,$$

and

$$Q = P_q \lim T_{m_j} \quad \left(\text{ or } Q = P_{q+1} \lim T_{m_j - h_{p(j)}} \right).$$

Iterating, taking into account the fact that $T_m \prod_{q \in M} P_q \rightarrow \Theta$ uniformly with respect to $m \in \mathbb{R}$ as $|M| \rightarrow \infty$, we stop as $Q = T_a \prod_{m \in M} P_m$.



9 Related Problems

1,2. Thouvenot's questions: Do mildly mixing rank-one transformations possess the properties *PID*? *MSJ*(2)?

3. Is $\text{Rank}(T^n)$ for mildly mixing rank-one transformation T equal to n ($n > 1$)?

4. King's question: will any ergodic self-joining of a rank-one transformation be a limit of off-diagonals measures?

5. Is it true that for any mildly mixing bounded construction T its symmetric powers $T^{\odot n}$ have simple spectrum? (see [8])

6,7. Let T be Chacon's map. Will the product $T \otimes T^2 \otimes T^3 \otimes T^4 \dots$ be of simple spectrum for Chacon's map ? for a mildly mixing bounded construction?

8. Consider a rank-one flow called rigid staircase flow. Fix $h_1 \in \mathbb{R}^+$ and a cut-sequence $r_j \rightarrow \infty$. Define spacers by

$$s_j(i) = \frac{i-1}{jr_j}, \quad i \leq r_j.$$

Will the corresponding rank-one flow be simple ? (see [4] or [7] for definitions of simplicity. An example of a rigid, simple transformation is given in [3].)

9. Let T, T' be rank-one constructions possessing the same spacer sequences, and $h_1 \neq h'_1$ for them. Are they disjoint ? (It is true for mixing T, T' , see arXiv:1109.0671.)

10. Are there mildly mixing (bounded) constructions T, T' such that T' is spectrally isomorphic to T but not isomorphic (as measure-preserving map) to T and T^{-1} ? For mixing rank-one flows this is possible: the flows T_t and $T_{\alpha t}$ ($\alpha > 1$) from arXiv:1002.2808 have the same spectrum (A. A. Prikhodko), but they are disjoint (see arXiv:1109.0671).

11. For rank-one transformations $\text{MSJ}(2) = \text{MSJ}$. Recall that MSJ can be defined as $\text{MSJ}(2) \cap \text{PID}$, where the property *PID* (pairwise independence determines the global independence) means that any pairwise independent self-joining have to be trivial (a product measure). This property has been introduced in [4] (see also [7]).

Let ν be a 3-fold self-joining of weakly mixing rank-one map T , and for all measurable $A, B \subset X$

$$\nu(A \times B \times X) = \nu(X \times A \times B) = \nu(A \times X \times B) = \mu(A)\mu(B).$$

Is it true that $\nu = \mu \times \mu \times \mu$?

12. Does the condition $T^{m_j}, T^{n_j}, T^{m_j-n_j} \rightarrow \Theta$ implies

$$\mu(A \cap T^{m_j} B \cap T^{n_j} C) \rightarrow \mu(A)\mu(B)\mu(C)$$

for a mildly mixing rank-one map T ?

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